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Engineering Notes

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Analysis of Formation Flying Control of a Pair of Nanosatellites

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Introduction

THIS Note analyzes and provides implementable solutions for stabilization and maneuver control of two nanosatellites subject to control and state constraints, bounded disturbance, and measurement error. This topic is currently an active research area (for example, see Refs. 1–6) using a variety of configurations and algorithms. There are very few methods for achieving constrained control. The approach we take is a modification of robust time-optimal control that reduces energy consumption. Published solutions of the constrained time-optimal control problem (e.g., Ref. 7) require computation of the robust controllability sets X_i. (X_i is the set of states that can be steered to a target set B^t in i time steps despite the disturbance.) In most practical applications, these sets are impossibly complex. In this paper, we avoid this problem by presenting a novel and effective method (the generalized diamond method) for computing inner approximations to the controllability sets. These inner approximations are then used to provide effective nonlinear control of the constrained system with little loss in performance.

Note that $\|\cdot\|$ denotes the Euclidean norm (2-norm) for a vector and the corresponding induced norm for a matrix and $\oplus$ denotes the Minkowski sum of two sets ($A \oplus B = \{a + b \mid a \in A,\ b \in B\}$).

Problem Description

The orbit and relative position of the two satellites is shown in Figs. 1a and 1b. For small distances d between the two satellites, for example, $d \leq 1000$ m, the Hill[8] equations of relative motion are

$$\ddot{x} = -2\omega\dot{y} + d_x + u_x, \qquad \ddot{y} = -2\omega\dot{x} + d_y + u_y$$

where (x, y) is the relative position of the following satellite, $\boldsymbol{d} = (d_x, d_y)$ the relative disturbance force, and $\boldsymbol{u} = (u_x, u_y)$ the actuator force used for control. [For computation, any vector such as $(\boldsymbol{x}, \boldsymbol{y})$, is regarded as a column vector.] The state-space equations for the relative motion of the slave satellite are

$$\dot{\boldsymbol{x}} = A_c\boldsymbol{x} + B_c(\boldsymbol{u} + \boldsymbol{d})$$

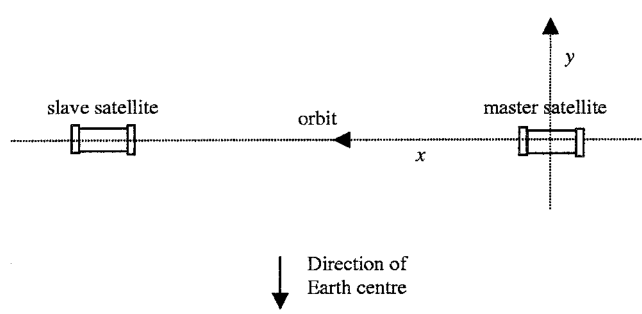
for $\boldsymbol{x} = (x, \dot{x}, y, \dot{y})$ and suitably chosen A_c and B_c, and $\omega = 0.001086$ rad/s. The corresponding discrete-time model is

$$\boldsymbol{x}_{k+1} = A\boldsymbol{x}_k + B\boldsymbol{u}_k + \xi_k$$

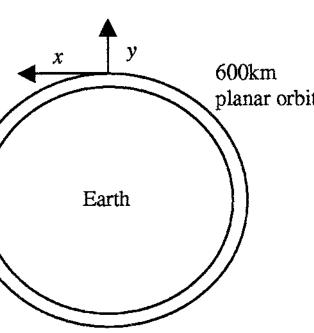
where T is the sampling period, $\boldsymbol{x}_k := \boldsymbol{x}(kT)$, and the control $u(t)$ has constant value u_k in the interval $[kT, (k+1)T)$

$$\xi_k = \int_{kT}^{(k+1)T} \exp[A_c(k+1)T - t]B_c\, d(t)\, \mathrm{d}t$$

the disturbance accumulated in the sampling period $[kT, (k+1)T)$. The control $\boldsymbol{u}$ is subject to the hard constraint $\boldsymbol{u} \in U$, where $U = \{(u_1, u_2) \| u_1| \leq u_m,\ |u_2| \leq u_m|\}$, where u_m is given and the state is required to stay in the set B^t. The controller must cope adequately with disturbances: inhomogeneity of the Earth's gravitational field, solar wind, and radiation pressure and gravitational forces of the sun and moon. The disturbance ξ that models disturbance forces as well as model inaccuracies is assumed to lie in a compact set D; a disturbance sequence $\{\xi_k\}$ is said to be admissible if $\xi_k \in D$ for all k. The control objectives are to steer the slave satellite from any initial state $\boldsymbol{x}_0$ in the set $\mathcal{X} := \{\boldsymbol{x} \mid \|\boldsymbol{x}\| \leq 1000\}$ to any specified relative position (x_r, y_r) in the target set B^t in as



a) Target position of the slave satellite at $(\boldsymbol{x}_r, \boldsymbol{y}_r)$



b) Close circular orbits of the two satellites

Fig. 1 Schematics of a simple formation flying configuration.

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short a time as possible while minimizing fuel consumption (maneuver control) and to maintain the position of the slave satellite thereafter at (x_r, y_r) with high accuracy, typically a few millimeters (stabilization).

Stabilization Control

We first consider the stabilization control problem: maintaining the state in the target set B^t despite the disturbance if the initial state is in B^t.

Admissible Solution

An admissible solution to this problem is a feedback law f_b: $B^t \to U$ such that the solution $\boldsymbol{x}_k$ of the closed-loop system

$$\boldsymbol{x}_{k+1} = A\boldsymbol{x}_k + Bf_b(\boldsymbol{x}_k) + \xi_k$$

satisfies $\boldsymbol{x}_k \in B^t$ for all $k \geq 0$ and all admissible disturbance sequences $\{\xi_k\}$ if the initial state $\boldsymbol{x}_0 \in B^t$. This is the case if the set B^t is robust control invariant, that is, if for all $\boldsymbol{x} \in B^t$ there exists a $\boldsymbol{u} \in U$ such that $A\boldsymbol{x} + B\boldsymbol{u} + \xi \in B^t$ for all $\xi \in D$. The following lemma provides an explicit method for checking whether or not an admissible solution to the stabilization problem exists.

Lemma: If A is invertible, a necessary and sufficient condition for the existence of an admissible feedback control law $f_b(\cdot)$ is

$$AB^t \subset (-BU) \oplus (B^t \ominus D)$$

Given the existence of an admissible feedback control law $f_b(\cdot)$, it is not necessary to determine it a priori. Rather, we propose to calculate, at each state $\boldsymbol{x}$ encountered, the control action $f_b(\boldsymbol{x})$; this is similar to model predictive control where, at each state $\boldsymbol{x}$ encountered, the control is determined by solving an optimal control problem. In our case, control u at any state $\boldsymbol{x} \in B^t$ may be found as follows. First, determine the set $U_s(\boldsymbol{x})$ defined by

$$U_s(\boldsymbol{x}) := \{\boldsymbol{u} \in U \mid A\boldsymbol{x} \oplus B\boldsymbol{u} \in B^t \ominus D\}$$

This set is not empty (because of the existence of an admissible control law). Next, determine that control $\boldsymbol{u} \in U_s(\boldsymbol{x})$ that minimizes some criterion, such as $\|\boldsymbol{u}\|$; the resultant control is $f_b(x)$ (the admissible control law evaluated at $\boldsymbol{x}$).

Example

As an example, consider the case when $D = \{\xi \mid \|\xi\| \leq 0.0025\}$, $B^t = (100, 0, 0, 0) \oplus \{\boldsymbol{x} \in \Re^4 \mid |x_i| \leq 0.01$ m, $i = 1, 3$, and $|x_i| \leq 0.01$ m/s, $i = 2, 4\}$, $T = 0.3$ s, and $u_m = 0.015$. The control was computed using the Geometric Bounding Toolbox[9] and the (simple) linear model as described earlier. Figure 2 shows 90 s of the orbit of an accurate nonlinear model (using control based on the linear model) that includes the effects of inhomogeneity of gravitational fields, solar wind, and radiation pressure. The nonlinear model of the orbit of each satellite was based on

$$\dot{\boldsymbol{r}} = \frac{-\mu\boldsymbol{r}}{r^3} - \frac{\partial \boldsymbol{B}}{\partial \boldsymbol{r}} + \boldsymbol{F}_{\text{sun}} + \boldsymbol{F}_{\text{moon}} + \boldsymbol{F}_{\text{spw}} \tag{1}$$

where $\boldsymbol{r}$ is the location vector of the center of mass of the satellite in the geocentric inertial coordinate system, $\boldsymbol{B}$ is the spherical harmonic expansion used to correct the gravitational potential for the Earth's nonsymmetric mass distribution (includes the J_2 term), $\boldsymbol{F}_{\text{sun}}$ and $\boldsymbol{F}_{\text{moon}}$ are the attraction forces of the sun and the moon, and $\boldsymbol{F}_{\text{spw}}$ are the forces due to solar wind and radiation pressure. These are modeled by random step disturbances bounded by 10 μN in the simulation. On a personal computer, the simulation took more than 90 s. The simulation included state measurement error with an upper bound of 0.0025.

Maneuver Control

We next consider the problem of steering any initial state in the set $\mathcal{X}$ to the target set B^t in minimum time despite the disturbance; this is the robust minimum-time control problem.

Controllability Sets X_i

The controllability set X_i is defined to be the set of states that can be steered (using a possibly time-varying control law $u(i) = f_m[x(i), i]$ to the target set B^t in i time steps or less for all admissible disturbance sequences. If A is invertible, these sets

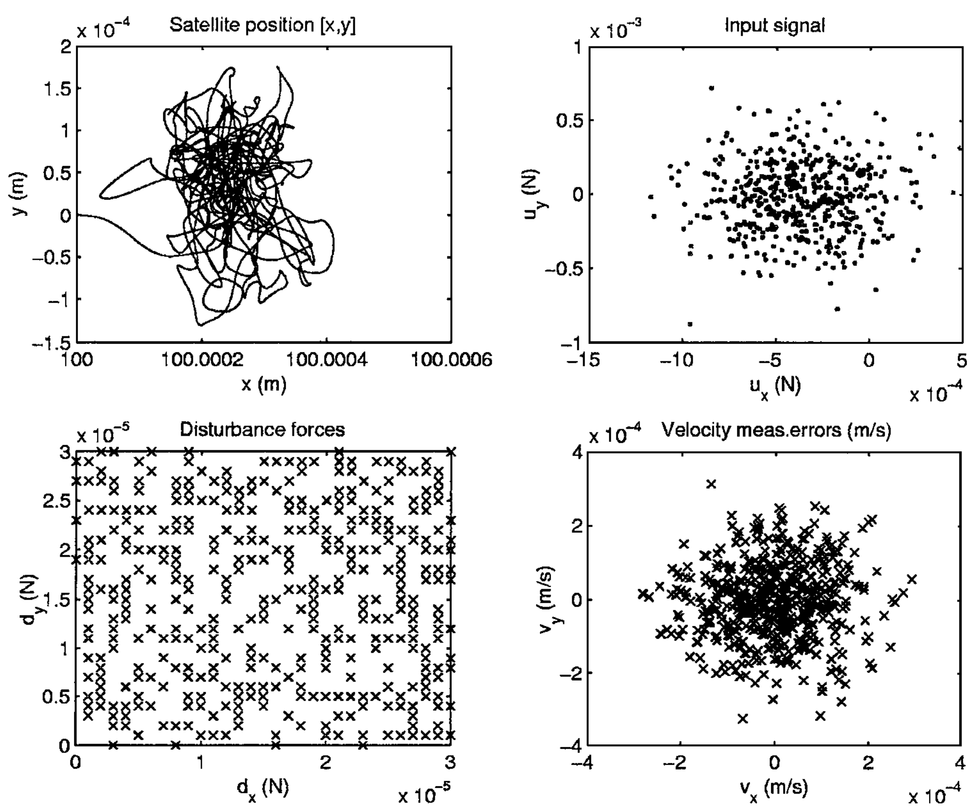

Fig. 2 Simulation of stabilization for 300 sampling periods.

may be computed recursively[4] as follows:

$$X_{i+1} = \{A^{-1}[(X_i \ominus D) \oplus (-A^{-1}BU)\}, \qquad i > 0$$

$$X_0 = B^t$$

A solution to the maneuver problem exists if there exists a finite time N such that $\mathcal{X} \subset X_N$. Given a sequence $\{X_i, i = 0, 1, \dots, N\}$, $\mathcal{X} \subset X_N$, a feedback solution to the constrained, robust time-opimal problem is easily obtained. If the current state is $\boldsymbol{x}$, find the minimum integer i such that $\boldsymbol{x} \in X_i$. Then find the polytope $U_m(\boldsymbol{x})$ defined by

$$U_m(\boldsymbol{x}) := \{\boldsymbol{u} \in U | A\boldsymbol{x} + B\boldsymbol{u} \in X_i \ominus D\}$$

It follows from the definition of controllability sets that $U_m(\boldsymbol{x})$ is not empty. Next determine that control action $u \in U_m(\boldsymbol{x}) \subset U$ that minimizes

$$V(\boldsymbol{u}) := \|A\boldsymbol{x} + B\boldsymbol{u} - \bar{\boldsymbol{x}}_i\|$$

where $\bar{\boldsymbol{x}}_i$ is the center of X_i. The solution $u^0(\boldsymbol{x})$ to this problem is the value $f_m(\boldsymbol{x})$ of the maneuver feedback control law $f_m(\cdot)$ at $\boldsymbol{x}$ $[f_m(\boldsymbol{x}) = u^0(\boldsymbol{x})]$. The problem of minimizing $V(\boldsymbol{u})$ subject to the constraint $\boldsymbol{u} \in U_m(\boldsymbol{x})$ is a quadratic program. The procedure yields robust time-optimal control; the control law $f_m(\cdot)$ steers any $\boldsymbol{x} \in X_i$ to B^t in i time steps or less for every admissible disturbance sequence.

Unfortunately, in most practical problems, the sets X_i become excessively complex as $i \to \infty$. Hence, we replace these sets by inner approximations $\hat{X}_i$, $i = 1, \dots, N$, for each i, $\hat{X}_i \subset X_i$. Therefore, it may be necessary to increase N to ensure $\mathcal{X} \subset \hat{X}_N$. The major consequence of this change is now that the control law $\hat{f}_m(\cdot)$ constructed using the inner approximations in place of the exact controllability sets now steers an $\boldsymbol{x} \in \hat{X}_i$ to B^t in i steps or less for every admissible disturbance sequence. Because $\hat{X} \subset X_i$, this time is possibly larger than the minimum possible. This is a small price to pay for the substantial decrease in complexity of the controllability sets.

Inner Approximation to a Polytope

Here we present the generalized diamond method, a novel and effective procedure for computing an inner approximation to any polytope. We define the diameter of a polytope as the line joining those two vertices of the polytope that maximize the distance between them:

$$\mathrm{diam}(P) := \mathrm{co}\{v, w\} = \{\mu v + (1-\mu)w | \mu \in [0, 1]\}$$

$$(v, w) := \arg \max_{v', w'} \{\|v' - w'\| \mid v', w' \in V(P)\}$$

where $V(P)$ is the set of vertices of the polytope P and $\mathrm{co}\{S\}$ denotes the convex hull of any set of points $S \subset \Re^n$. Given a polytope P in $\Re^n$, let $H(P)$ denote the subspace such that $P \subset \boldsymbol{x} + H(P)$ for some $\boldsymbol{x} \in \Re^n$; the dimension of $P[\dim(P)]$ is defined to the dimension of $H(P)$.

The algorithm to compute diamond approximation of polytope P is as follows.

0) Find a diameter (v_1, w_1) of P. Set $P_1 = \mathrm{co}\{v_1, w_1\}$. Determine L_1, the subspace of $\Re^n$ orthogonal to $H(P_1)$.

1) At iteration $i \geq 2$ of the algorithm, project all of the vertices of P except those in P_{i-1}, onto L_{i-1}, to obtain the polytope S_i. Obtain a diameter $(\bar{v}_i, \bar{w}_i)$ of S_i. If $\bar{v}_i = \mathrm{proj}_{L_{i-1}}(v_i)$ and $\bar{w}_i = \mathrm{proj}_{L_{i-1}}(w_i)$, set $P_i = \mathrm{co}\{P_{i-1}, v_i, w_i\}$ and L_i the subspace of $\Re^n$ orthogonal to P_i.

2) If $i = n$, stop. Else set $i = i + 1$ and go to step 1.

The algorithm generates a polytope $\hat{P}$ with $2n$ vertices and implicitly defines the map $P \mapsto \hat{P}$, $\hat{P} = \mathrm{diamond}(P)$. Note that, if P is n dimensional, then P_i will be i dimensional, L_i will be $n - i$ dimensional for $i = 1, \dots, n$, and in particular $L_n = \{0\}$ is the zero space.

Approximate Controllability Sets

The generalized diamond procedure may be used to compute approximate controllability sets $\hat{X}_i$ via the recursion

$$\hat{X}_{i+1} = \mathrm{diamond}\left(\left\{A^{-1}(\hat{X}_i \ominus D) \oplus (-A^{-1}BU)\right\}\right), \qquad i > 0$$

$$\hat{X}_0 = B^t$$

Example

We illustrate the procedure on the example employed earlier. The approximate controllability sets $\hat{X}_1, \dots, \hat{X}_{200}$ were generated, using the generalized diamond algorithm, in 1 min on an average

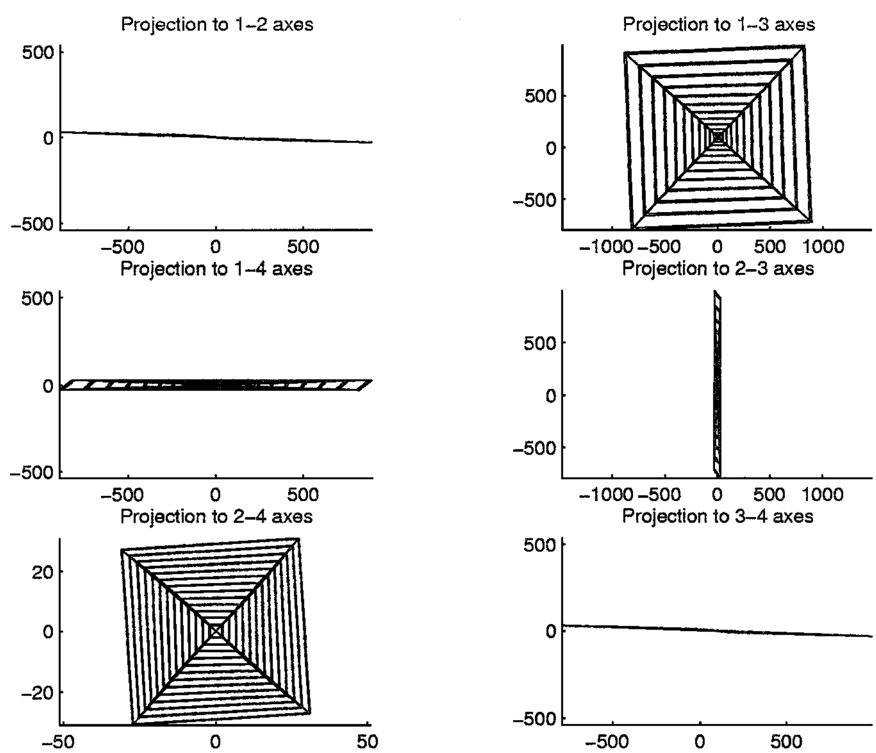

Fig. 3 Wire-frame views of the internal approximation to the set X_{200}.

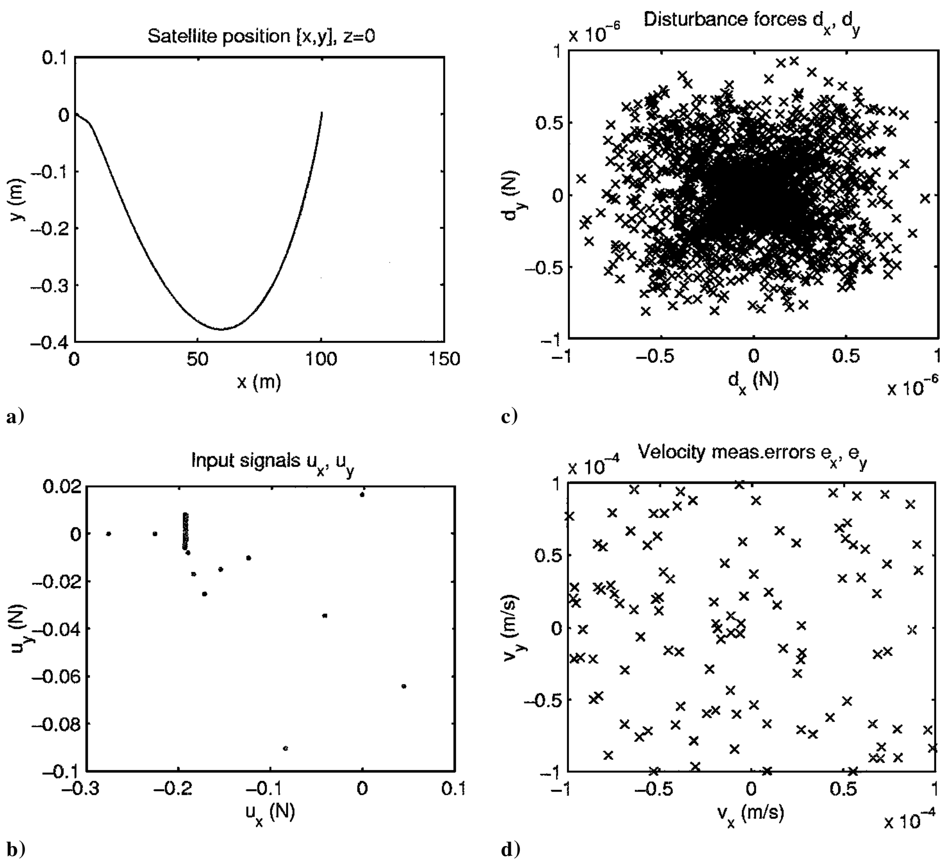

Fig. 4 Results of the three-dimensional nonlinear simulation projected onto the x, y plane of orbit.

Table 1 Illustrative CPU times and number of vertices for the generalized diamond-based and the exact-polytope-based controllable set computations

	Generalized diamond based		Exact polytope based[a]	
X_i index	CPU time 1	Vertices 1	CPU time 2	Vertices 2
1	0	16	0	46
2	2.223	16	164.687	214
3	4.016	16	589.131	812
4	6.2090	16	1723.567	1714
5	7.8510	16	4611.781	3012

[a]Synchronized after X_1.

personal computer using MATLAB® version 7.1 (using the Geometric Bounding Toolbox[9]). The polytope $\hat{X}_{200}$ is shown in Fig. 3. The least integer ℓ such that the initial state $x_0 = (0, 6, 0, 0)$ lies in $\hat{X}_\ell$ is 187. (It requires 187 sampling periods to steer $\boldsymbol{x}_0$ to B^t robustly.) A simulation of maneuver control (using an accurate nonlinear model) is shown in Fig. 4. Figure 4a shows that the controller $f_m(\cdot)$ transfers the initial state (0, 6, 0, 0) to the target state (100, 0, 0, 0) fairly directly.

To show the (dramatic) difference between the use of the approximate and exact controllability sets, the complexity of and computation time required to determine $\{X_1, \ldots, X_5\}$ and $\{\hat{X}_1, \ldots, \hat{X}_5\}$ are shown in Table 1. The diamond method produces approximate polytopes with a low number of vertices (16) and requires low computation times; complexity and computation time of the exact polytopes X_i increase rapidly with i.

Although there can be a large difference in the size of X_n and its approximation $\hat{X}_n$, the diamond method is, unlike the exact method, numerically feasible.

Conclusions

This paper describes the generalized diamond method for approximating complex polytopes in the state-space of satellite orbital dynamics and shows how this procedure may be used to compute approximate controllability sets and how these sets may, in turn, be used to obtain a nonlinear controller that robustly steers any initial state to a specified target set and thereafter maintains the state in the target set despite the disturbance. This procedure has considerable advantages over the use of exact controllability sets that are usually excessively complex. A simplified model was used to compute the controller, whereas a more realistic noninear model was used for simulation. It would be desirable to consider next the coupled orbit and satellite control problem arising in formation flying.

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